1) *Find the inverse of the following matrix employing elementary transformations:*

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution. The given matrix is $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to \frac{R_1}{3} \\ A \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} A \\ R_2 \to R_2 - 2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} A \\ R_2 \to -R_2 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} \begin{matrix} A \\ R_3 \to R_3 + R_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} \begin{matrix} A \\ R_3 \to -3R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad \begin{matrix} R_1 \to R_1 - \frac{4}{3}R_3 \\ R_2 \to R_2 + \frac{4}{3}R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \quad R_1 \to R_1 + R_2$$

Hence, $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ Ans.

2)

$$A = IA$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -13 & 5 & 1 \end{bmatrix} A$$

$R_3 \to \frac{-1}{2} R_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -13/2 & 5/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 - R_3, \quad R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15/2 & -5/2 & -1/2 \\ 9/2 & -3/2 & -1/2 \\ -13/2 & 5/2 & 1/2 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 & 1/2 \\ 9/2 & -3/2 & -1/2 \\ -13/2 & 5/2 & 1/2 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} -3/2 & 1/2 & 1/2 \\ 9/2 & -3/2 & -1/2 \\ -13/2 & 5/2 & 1/2 \end{bmatrix}$$

3) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

Minor of order $3 = |A| = 0$

Minor of order 2

$$\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow$ $\rho(A) = 2 \neq$ No. of vectors

$\Rightarrow$ vectors are Linearly Dependent

4) let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 3 & 1 & 4 & 6 \end{bmatrix}$.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 \neq$ No. of vectors

$\Rightarrow$ vectors are L.D.

5) $X_1 = (3, -2, 4)$

$X_2 = (1, 0, 2)$

$X_3 = (1, -1, 1)$

Let $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

Minor of order $3 = |A| = 0$.

$\rho(A) \neq 3 =$ No. of vectors

$\Rightarrow$ Vectors are L.D.

Relation: Let $k_1X_1 + k_2X_2 + k_3X_3 = 0$

$k_1(3, -2, 4) + k_2(1, 0, 2) + k_3(1, -1, 1) = 0$

$(3k_1 + k_2 + k_3,\ -2k_1 - k_3,\ 4k_1 + 2k_2 + k_3) = (0, 0, 0)$

On comparing

$3k_1 + k_2 + k_3 = 0$

$-2k_1 - k_3 = 0$

$4k_1 + 2k_2 + k_3 = 0$

On solving, we get

$k_1 = 1, \quad k_2 = -1, \quad k_3 = 2.$

$\therefore$ $\boxed{X_1 - X_2 + 2X_3 = 0}$ is required Relation.

6) vectors are L.D

$\rho(A) \neq$ No. of vectors

$\rho(A) \neq 3$

$\Rightarrow |A| = 0$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ k & 5 & 6 \end{vmatrix} = 0.$$

$$\boxed{k = \frac{20}{11}}$$